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AN OBJECTIVE KINEMATICAL FORMALISM FOR THE MODELING OF ELASTIC–PLASTIC MATERIALS SUBJECT TO LARGE DEFORMATION

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Abstract—A kinematical formalism for the analysis of two superposed finite elastic–plastic deformations is presented. This is offered as an alternative to the unnecessarily complex (in the view of this author) kinematical development of Schieck and Stumpf in the accompanying Journal article entitled "Deformation Analysis for Finite Elastic–Plastic Strains in a Lagrangean-Type Description" (Schieck and Stumpf, 1993, *Int. J. Solids Structures* **30**). It is claimed that this provides an equivalent kinematical foundation for their principal results.

1. INTRODUCTION

Schieck and Stumpf (1993) present a theoretical framework for the analysis of structural deformation comprised of arbitrarily large elastic and plastic strains. Their kinematical treatment of this problem, which is quite formal in nature, is based on the superposition of two sequential finite elastic–plastic deformations. They state that certain numerical efficiencies can be achieved by adopting a scheme whereby the first of these two deformations is periodically updated so that the second deformation conforms to the description "moderately large". It is explicitly stated after their eqn (83) that "essential simplifications" are possible for materials which admit a quadratic energy function in logarithmic (Hencky) strain for elastic stretches in the range of 0.7 to 1.3.

It is important to realize that their theory extends only to materials whose elastic properties are both initially isotropic and invariant under continued plastic flow. Anisotropic behavior is strictly limited to the plastically induced effect of kinematic hardening which, together with the commonly understood mechanism of isotropic hardening, influences only the yield criteria and the plastic flow rule. This was not emphasized in an earlier draft of this paper. Moreover, the complexity of the kinematical formulation, particularly as it regards the various orthogonal rotation tensors, suggested (to me) that some attempt was being made to map the orientation of an anisotropic elastic structure. There now seems to be agreement that this would not be possible within a theory of this type.†

In view of the restricted nature of this theory (as delimited above), it is not at all clear (and in fact somewhat unsettling) that such a complex kinematical formulation is required—even for two superposed finite strain elastic–plastic deformations. To appreciate the complexity of the kinematics one need only glance at their Fig. 2 which represents only a partial illustration of the various configurations to which they refer.

In this note, I have sketched a much simpler kinematical formulation for this problem involving only a bare minimum of *objective* kinematical variables, which include only a single orthogonal rotation tensor. I contend that this is all that is needed in view of the absence of structural isotropy as discussed above. Moreover, this formulation is similarly amenable to the introduction of an elastic log (Hencky) strain tensor and would also benefit from "essential simplifications" if the material admits a quadratic energy function. I believe that this formulation effectively parallels theirs and would serve as an equivalent basis for their incremental approximation procedures of which I have no criticism.

† See Dashner (1986a) and the ensuing commentary by Casey (1987) for a more complete discussion of this matter.

2. ALTERNATIVE KINEMATICS

An adequate kinematical description of the Lagrangean-type for the mechanical behavior of an elastic–plastic solid whose elastic properties are both initially isotropic and invariant under continued plastic flow begins with the selection of a fixed reference configuration $\mathscr{B}_0$ corresponding to some initial unstressed state of the material. It shall be assumed that the elastic energy and stress response for this material, prior to the onset of any plastic deformation, are given by the response functions

$$\Psi = \hat{\Psi}(\mathbf{C}), \quad \mathbf{C} \equiv \mathbf{F}^{\mathrm{T}}\mathbf{F},$$
$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{C}) = \frac{\partial \hat{\Psi}}{\partial \mathbf{E}} = 2\frac{\partial \hat{\Psi}}{\partial \mathbf{C}}, \quad 2\mathbf{E} \equiv \mathbf{C} - \mathbf{I}, \tag{1}$$

in terms of the elastic strain energy per unit reference volume Ψ, the symmetric Piola–Kirchhoff stress tensor $\mathbf{S}$, and the total (at this point completely elastic) Green deformation and strain tensors $\mathbf{C}$ and $\mathbf{E}$. All Lagrangean measures are referred to the fixed reference configuration $\mathscr{B}_0$ and the scalar and tensor functions $\hat{\Psi}$ and $\hat{\mathbf{S}}$ are assumed to be isotropic to insure isotropic elastic response. As outlined in the Appendix and explicitly stated in (A16), the subsequent energy and stress response for this elastic–plastic continua is determined by the forms

$$\Psi = \hat{\Psi}(\mathbf{F}_e^{\mathrm{T}}\mathbf{F}_e),$$
$$\mathbf{S} = \mathbf{F}_p^{-1}\hat{\mathbf{S}}(\mathbf{F}_e^{\mathrm{T}}\mathbf{F}_e)\mathbf{F}_p^{-\mathrm{T}}, \tag{2}$$

expressed in terms of the standard elastic–plastic gradient decomposition constituents

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p \tag{3}$$

based on the usual notion of an unstressed intermediate configuration. For completeness, note that the combined effect of simultaneous proper orthogonal rotations of the current, intermediate (unstressed) and reference configurations, respectively given by $\mathbf{Q}$, $\mathbf{M}$ and $\mathbf{P}$, is specified by the transformations

$$\begin{aligned}
\mathbf{F} = \mathbf{R}\mathbf{U} &\to \mathbf{Q}\mathbf{F}\mathbf{P}^{\mathrm{T}},\\
\mathbf{R} &\to \mathbf{Q}\mathbf{R}\mathbf{P}^{\mathrm{T}},\\
\mathbf{U} &\to \mathbf{P}\mathbf{U}\mathbf{P}^{\mathrm{T}},\\
\mathbf{F}_e = \mathbf{R}_e\mathbf{U}_e &\to \mathbf{Q}\mathbf{F}_e\mathbf{M}^{\mathrm{T}},\\
\mathbf{R}_e &\to \mathbf{Q}\mathbf{R}_e\mathbf{M}^{\mathrm{T}},\\
\mathbf{U}_e &\to \mathbf{M}\mathbf{U}_e\mathbf{M}^{\mathrm{T}},\\
\mathbf{F}_p = \mathbf{R}_p\mathbf{U}_p &\to \mathbf{M}\mathbf{F}_p\mathbf{P}^{\mathrm{T}},\\
\mathbf{R}_p &\to \mathbf{M}\mathbf{R}_p\mathbf{P}^{\mathrm{T}},\\
\mathbf{U}_p &\to \mathbf{P}\mathbf{U}_p\mathbf{P}^{\mathrm{T}},\\
\Psi &\to \Psi,\\
\mathbf{S} &\to \mathbf{P}\mathbf{S}\mathbf{P}^{\mathrm{T}}.
\end{aligned} \tag{4}$$

From these it is a simple matter to confirm that the above constitutive relations are *objective*, i.e. *frame invariant* (in that they are insensitive to arbitrary specification of $\mathbf{Q}$), insensitive to an arbitrary reorientation $\mathbf{M}$ of the unstressed configuration (this due to the isotropy of the response functions $\hat{\Psi}$ and $\hat{\mathbf{S}}$), and fully isotropic since any prerotation $\mathbf{P}$ of the reference configuration serves only to rotate the Kirchhoff stress tensor $\mathbf{S}$ in an identical manner. The particular observation as to the causal insignificance of the orientation of the intermediate unstressed configuration (arbitrary specification of $\mathbf{M}$) is clearly connected to the isotropic nature of the elastic response and is in no manner connected to any notion of *objectivity* or *invariance of frame*. This concept is the key to relative efficiency of the present kinematical formalism.

It shall now be assumed that at a given instant in time $t = t_1 > 0$, the state of elastic–plastic deformation is adequately described by the gradient decomposition

$$\mathbf{F}_1 = \mathbf{F}_{e_1}\mathbf{F}_{p_1}. \tag{5}$$

Owing to invariance under arbitrary specification of orthogonal $\mathbf{M}$ in (4), it is a simple matter to rewrite the time $t = t_1$ gradient in terms of an equivalent kinematically compatible decomposition, i.e.

$$\begin{aligned}
\mathbf{F}_1 = \bar{\mathbf{F}}_e\bar{\mathbf{F}}_p &= \mathbf{F}_{e_1}\mathbf{F}_{p_1},\\
(\bar{\mathbf{F}}_e)(\bar{\mathbf{F}}_p) &= (\mathbf{F}_{e_1}\mathbf{M}^T)(\mathbf{M}\mathbf{F}_{p_1})\\
&= (\mathbf{R}_{e_1}\mathbf{U}_{e_1}\mathbf{M}^T)(\mathbf{M}\mathbf{R}_{p_1}\mathbf{U}_{p_1})\\
&= (\mathbf{R}_{e_1}\mathbf{U}_{e_1}\mathbf{R}_{p_1})(\mathbf{U}_{p_1}), \quad \mathbf{M} = \mathbf{R}_{p_1}^T\\
&= [\mathbf{R}_{e_1}(\mathbf{R}_{p_1}\mathbf{R}_{p_1}^T)\mathbf{U}_{e_1}\mathbf{R}_{p_1}](\mathbf{U}_{p_1})\\
&= [(\mathbf{R}_{e_1}\mathbf{R}_{p_1})(\mathbf{R}_{p_1}^T\mathbf{U}_{e_1}\mathbf{R}_{p_1})](\mathbf{U}_{p_1})\\
(\bar{\mathbf{F}}_e)(\bar{\mathbf{F}}_p) &= (\bar{\mathbf{R}}_e\bar{\mathbf{U}}_e)(\bar{\mathbf{U}}_p)
\end{aligned} \tag{6}$$

$$\Rightarrow \begin{cases} \bar{\mathbf{F}}_e = \bar{\mathbf{R}}_e\bar{\mathbf{U}}_e, \quad \bar{\mathbf{R}}_e = \mathbf{R}_{e_1}\mathbf{R}_{p_1} \quad \text{and} \quad \bar{\mathbf{U}}_e = \mathbf{R}_{p_1}^T\mathbf{U}_{e_1}\mathbf{R}_{p_1},\\ \bar{\mathbf{F}}_p = \bar{\mathbf{U}}_p, \quad \bar{\mathbf{R}}_p = \mathbf{I} \quad \text{and} \quad \bar{\mathbf{U}}_p = \mathbf{U}_{p_1}. \end{cases}$$

Having thus obtained the elastic and plastic stretch tensors $\bar{\mathbf{U}}_e$ and $\bar{\mathbf{U}}_p$ we choose to define the $t = \bar{t} = t_1$ *base state* for this material element through the elastic–plastic deformation

$$\bar{\mathbf{F}} = \bar{\mathbf{R}}_e^T\mathbf{F}_1 = \bar{\mathbf{U}}_e\bar{\mathbf{U}}_p \tag{7}$$

differing from the actual material state at $t = t_1$ only by a local superposed rigid body rotation of the current configuration.

It shall further be assumed that at the current instant, $t > \bar{t}$, kinematically compatible elastic–plastic deformation gradient components are given by

$$\begin{aligned}
\mathbf{F} &= \overset{+}{\mathbf{F}}\bar{\mathbf{U}}_e\bar{\mathbf{U}}_p, \quad \overset{+}{\mathbf{F}} = \overset{+}{\mathbf{R}}\overset{+}{\mathbf{U}},\\
\mathbf{F}_p &= \overset{+}{\mathbf{U}}_p\bar{\mathbf{U}}_p,\\
\mathbf{F}_e &= \overset{+}{\mathbf{F}}\bar{\mathbf{U}}_e\overset{+}{\mathbf{U}}{}_p^{-1}.
\end{aligned} \tag{8}\dagger$$

Clearly, $\overset{+}{\mathbf{F}} = \overset{+}{\mathbf{R}}\overset{+}{\mathbf{U}}$ defines the current element configuration relative to the element configuration $\bar{\mathbf{F}} = \bar{\mathbf{U}}_e\bar{\mathbf{U}}_p$ associated with the *base state* with $\overset{+}{\mathbf{U}}$ giving the incremental material

† This form for the plastic gradient component $\mathbf{F}_p$ is always possible due to the causal insignificance of the orientation of the intermediate unstressed configuration. Recall that the constitutive model is completely insensitive to arbitrary specification of the orthogonal rotation tensor $\mathbf{M}$.

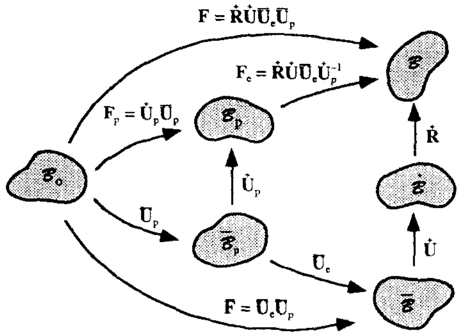

Fig. 1. State configuration space for two superposed elastic–plastic deformations.

stretch since $t = \bar{t}$. Similarly, $\overset{+}{\mathbf{U}}_p$ is seen to determine the incremental plastic stretch since $t = \bar{t}$. These kinematically compatible configurations are illustrated in Fig. 1.

Substitution into the constitutive forms (2) yields the response functions

$$\Psi = \hat{\Psi}(\overset{+}{\mathbf{U}}_p^{-1}\bar{\mathbf{U}}_e\overset{+}{\mathbf{C}}\bar{\mathbf{U}}_e\overset{+}{\mathbf{U}}_p^{-1}), \quad \overset{+}{\mathbf{C}} \equiv \overset{+}{\mathbf{U}}^2,$$
$$\mathbf{S} = \bar{\mathbf{U}}_p^{-1}\{\overset{+}{\mathbf{U}}_p^{-1}[\hat{\mathbf{S}}(\overset{+}{\mathbf{U}}_p^{-1}\bar{\mathbf{U}}_e\overset{+}{\mathbf{C}}\bar{\mathbf{U}}_e\overset{+}{\mathbf{U}}_p^{-1})]\overset{+}{\mathbf{U}}_p^{-1}\}\bar{\mathbf{U}}_p^{-1}, \tag{9}$$

written exclusively in terms of *objective*, symmetric stretch and deformation tensors.

Recalling that the *base state* variables $\bar{\mathbf{U}}_p$ and $\bar{\mathbf{U}}_e$ are known and fixed, it is evident that a complete theory requires rate equations with which to increment the values of $\overset{+}{\mathbf{U}}$ and $\overset{+}{\mathbf{U}}_p$ during an ongoing deformation process. Since this theory falls within the purview of the original Green–Naghdi (1965) formulation, it shall initially be assumed that there exist objective rate equations for the total Green strain $\mathbf{E}$ and the total plastic strain $\mathbf{E}_p$. It is easily seen that the existence of such rate forms leads to equivalent rate expressions for our incremental stretches $\overset{+}{\mathbf{U}}$ and $\overset{+}{\mathbf{U}}_p$. In the derivation of these forms, use is made of the fourth order "symmetric product" and "bracket" tensors

$$\mathbb{S}_\mathbf{A} \cdot \mathbf{X} \equiv \tfrac{1}{2}(\mathbf{AX}+\mathbf{XA}),$$
$$\mathbb{B}_\mathbf{A} \cdot \mathbf{X} \equiv \mathbf{AXA}, \tag{10}†$$

corresponding to a given symmetric tensor $\mathbf{A}$ and defined on the set of all symmetric tensors $\mathbf{X}$. The algebraic properties of these "double tensors" as well as their connection to log-strain type variables and small strain expansions are detailed in the Appendix of Dashner (1990).

Starting with the plastic strain expression

$$\begin{aligned} 2\mathbf{E}_p &= \mathbf{C}_p - \mathbf{I} \\ &= \mathbf{F}_p^T\mathbf{F}_p - \mathbf{I} \\ &= \bar{\mathbf{U}}_p\overset{+}{\mathbf{U}}{}_p^2\bar{\mathbf{U}}_p - \mathbf{I} \end{aligned} \tag{11}$$

it follows that

† Of particular interest here is the fact that $\mathbb{S}_\mathbf{A}$ is non-singular for symmetric, definite symmetric $\mathbf{A}$, and $\mathbb{B}_\mathbf{A}$ is non-singular for symmetric, non-singular symmetric $\mathbf{A}$.

$$2\dot{\mathbf{E}}_p = \bar{\mathbf{U}}_p\left[\overset{+}{\mathbf{U}}_p\frac{d\overset{+}{\mathbf{U}}_p}{dt} + \frac{d\overset{+}{\mathbf{U}}_p}{dt}\overset{+}{\mathbf{U}}_p\right]\bar{\mathbf{U}}_p,$$

$$\dot{\mathbf{E}}_p = \mathbb{B}_{\bar{\mathbf{U}}_p}\circ\mathbb{S}_{\overset{+}{\mathbf{U}}_p}\cdot\frac{d\overset{+}{\mathbf{U}}_p}{dt} \Leftrightarrow \frac{d\overset{+}{\mathbf{U}}_p}{dt} = \mathbb{S}^{-1}_{\overset{+}{\mathbf{U}}_p}\circ\mathbb{B}^{-1}_{\bar{\mathbf{U}}_p}\cdot\dot{\mathbf{E}}_p \tag{12}$$

and similarly

$$\begin{aligned}
2\mathbf{E} &= \mathbf{C}-\mathbf{I}\\
&= \mathbf{F}^{\mathrm{T}}\mathbf{F}-\mathbf{I}\\
&= \bar{\mathbf{U}}_p\bar{\mathbf{U}}_e\overset{+}{\mathbf{U}}{}^2\bar{\mathbf{U}}_e\bar{\mathbf{U}}_p-\mathbf{I},
\end{aligned}$$

$$2\dot{\mathbf{E}} = \bar{\mathbf{U}}_p\bar{\mathbf{U}}_e\left[\overset{+}{\mathbf{U}}\frac{d\overset{+}{\mathbf{U}}}{dt} + \frac{d\overset{+}{\mathbf{U}}}{dt}\overset{+}{\mathbf{U}}\right]\bar{\mathbf{U}}_e\bar{\mathbf{U}}_p,$$

$$\dot{\mathbf{E}} = \mathbb{B}_{\bar{\mathbf{U}}_p}\circ\mathbb{B}_{\bar{\mathbf{U}}_e}\circ\mathbb{S}_{\overset{+}{\mathbf{U}}}\cdot\frac{d\overset{+}{\mathbf{U}}}{dt} \Leftrightarrow \frac{d\overset{+}{\mathbf{U}}}{dt} = \mathbb{S}^{-1}_{\overset{+}{\mathbf{U}}}\circ\mathbb{B}^{-1}_{\bar{\mathbf{U}}_e}\circ\mathbb{B}^{-1}_{\bar{\mathbf{U}}_p}\cdot\dot{\mathbf{E}}. \tag{13}$$

Finally, in an iterative procedure in which the strains associated with the incremental stretches $\overset{+}{\mathbf{U}}$ and $\overset{+}{\mathbf{U}}_p$, or with the incremental elastic deformation related to

$$\overset{+}{\mathbf{C}}_e \equiv \overset{+}{\mathbf{U}}{}_p^{-1}\bar{\mathbf{U}}_e\overset{+}{\mathbf{C}}\bar{\mathbf{U}}_e\overset{+}{\mathbf{U}}{}_p^{-1}, \tag{14}$$

are allowed to "grow" only to a certain limit before defining a new *base state* corresponding to the present state, update relations for $\bar{\mathbf{U}}_p$ and $\bar{\mathbf{U}}_e$, and initial conditions for $\overset{+}{\mathbf{U}}$ and $\overset{+}{\mathbf{U}}_p$ during the next "grow phase" are required. With reference to eqns (6) and (7), this is easily achieved since

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p \quad \text{with} \quad \begin{cases} \mathbf{F}_p = \overset{+}{\mathbf{U}}_p\bar{\mathbf{U}}_p, \\ \mathbf{F}_e = \overset{+}{\mathbf{R}}\overset{+}{\mathbf{U}}\bar{\mathbf{U}}_e\overset{+}{\mathbf{U}}{}_p^{-1}, \end{cases} \tag{15}$$

is a kinematically compatible gradient decomposition for the present state. From these relations it is easily seen that

$$\mathbf{F} = \bar{\mathbf{F}}_e^{\mathrm{new}}\bar{\mathbf{F}}_p^{\mathrm{new}} = (\bar{\mathbf{R}}_e^{\mathrm{new}}\bar{\mathbf{U}}_e^{\mathrm{new}})(\bar{\mathbf{U}}_p^{\mathrm{new}}) = \mathbf{F}_e\mathbf{F}_p,$$

$$\mathbf{F}_p = \mathbf{R}_p\mathbf{U}_p = \overset{+}{\mathbf{U}}_p\bar{\mathbf{U}}_p \begin{cases} \mathbf{U}_p = \sqrt{\bar{\mathbf{U}}_p\overset{+}{\mathbf{U}}{}_p^2\bar{\mathbf{U}}_p} \\ \mathbf{R}_p = \overset{+}{\mathbf{U}}_p\bar{\mathbf{U}}_p\mathbf{U}_p^{-1}, \end{cases}$$

$$\mathbf{F}_e = \mathbf{R}_e\mathbf{U}_e = \overset{+}{\mathbf{R}}\overset{+}{\mathbf{U}}\bar{\mathbf{U}}_e\overset{+}{\mathbf{U}}{}_p^{-1} \begin{cases} \mathbf{U}_e = \sqrt{\overset{+}{\mathbf{U}}{}_p^{-1}\bar{\mathbf{U}}_e\overset{+}{\mathbf{U}}{}^2\bar{\mathbf{U}}_e\overset{+}{\mathbf{U}}{}_p^{-1}}, \\ \mathbf{R}_e = \overset{+}{\mathbf{R}}\overset{+}{\mathbf{U}}\bar{\mathbf{U}}_e\overset{+}{\mathbf{U}}{}_p^{-1}\mathbf{U}_e^{-1}, \end{cases}$$

$$\Rightarrow \begin{cases} \bar{\mathbf{U}}_p^{\mathrm{new}} = \mathbf{U}_p \\ \bar{\mathbf{U}}_e^{\mathrm{new}} = \mathbf{R}_p^{\mathrm{T}}\mathbf{U}_e\mathbf{R}_p \\ \bar{\mathbf{R}}_e^{\mathrm{new}} = \mathbf{R}_e\mathbf{R}_p \end{cases} \quad \text{and} \quad \begin{array}{l} \overset{+}{\mathbf{U}}_p|_o = \overset{+}{\mathbf{U}}|_o = \mathbf{I}, \\ \overset{+}{\mathbf{R}}|_o = \bar{\mathbf{R}}_e^{\mathrm{new}}. \end{array} \tag{16}$$

In the above relations, observe that this updating procedure "zeros" all incremental strains associated with the stretch variables $\overset{+}{\mathbf{U}}$ and $\overset{+}{\mathbf{U}}_{\mathrm{p}}$. Note also that while the associated rotation variable $\overset{+}{\mathbf{R}}$ is not "zeroed" its significance is of limited importance. In fact, $\overset{+}{\mathbf{R}}$ is useful only for relating Lagrangian measures to their Eulerian counterparts defined over the current element configuration.

3. LOG-STRAIN FORMS

As stated by Schieck and Stumpf, a quadratic energy function expressed in terms of an elastic log (Hencky) strain tensor does lead to certain simplifications. For this, note that (8) and $(9)_2$ lead to the expression

$$\boldsymbol{\tau} = \mathbf{FSF}^{\mathrm{T}} = \overset{+}{\mathbf{R}}\overset{+}{\mathbf{U}}\bar{\mathbf{U}}_{\mathrm{e}}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}[\hat{\mathbf{S}}(\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}\bar{\mathbf{U}}_{\mathrm{e}}\overset{+}{\mathbf{C}}\bar{\mathbf{U}}_{\mathrm{e}}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1})]\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}\bar{\mathbf{U}}_{\mathrm{e}}\overset{+}{\mathbf{U}}\overset{+}{\mathbf{R}}^{\mathrm{T}}$$

$$= \overset{+}{\mathbf{R}}\overset{+}{\mathbf{F}}_{\mathrm{e}}[\hat{\mathbf{S}}(\overset{+}{\mathbf{F}}{}_{\mathrm{e}}^{\mathrm{T}}\overset{+}{\mathbf{F}}_{\mathrm{e}})]\overset{+}{\mathbf{F}}{}_{\mathrm{e}}^{\mathrm{T}}\overset{+}{\mathbf{R}}^{\mathrm{T}}, \quad \overset{+}{\mathbf{F}}_{\mathrm{e}} \equiv \overset{+}{\mathbf{U}}\bar{\mathbf{U}}_{\mathrm{e}}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1} \tag{17}$$

for the Kirchhoff stress tensor $\boldsymbol{\tau} = (\rho_0/\rho)\boldsymbol{\sigma}$. Thus, owing to the isotropy of the stress response function, the back-rotated (to configuration $\overset{+}{\mathscr{B}}$) Kirchhoff stress tensor is given by

$$\overset{+}{\boldsymbol{\tau}} \equiv \overset{+}{\mathbf{R}}{}^{\mathrm{T}}\boldsymbol{\tau}\overset{+}{\mathbf{R}} = \overset{+}{\mathbf{V}}_{\mathrm{e}}[\hat{\mathbf{S}}(\overset{+}{\mathbf{V}}{}_{\mathrm{e}}^{2})]\overset{+}{\mathbf{V}}_{\mathrm{e}}, \quad \overset{+}{\mathbf{V}}_{\mathrm{e}} \equiv \sqrt{\overset{+}{\mathbf{F}}_{\mathrm{e}}\overset{+}{\mathbf{F}}{}_{\mathrm{e}}^{\mathrm{T}}} \tag{18}$$

in terms of the left stretch tensor $\overset{+}{\mathbf{V}}_{\mathrm{e}}$ associated with

$$\overset{+}{\mathbf{F}}_{\mathrm{e}} = \overset{+}{\mathbf{R}}_{\mathrm{e}}\overset{+}{\mathbf{U}}_{\mathrm{e}} = \overset{+}{\mathbf{V}}_{\mathrm{e}}\overset{+}{\mathbf{R}}_{\mathrm{e}} = \overset{+}{\mathbf{U}}\bar{\mathbf{U}}_{\mathrm{e}}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}. \tag{19}$$

With reference to (1), (9) and the fourth order "symmetrizer" (10), it is evident that

$$\Psi = \hat{\Psi}(\overset{+}{\mathbf{B}}_{\mathrm{e}}), \quad \overset{+}{\mathbf{B}}_{\mathrm{e}} \equiv \overset{+}{\mathbf{V}}{}_{\mathrm{e}}^{2} = \overset{+}{\mathbf{F}}_{\mathrm{e}}\overset{+}{\mathbf{F}}{}_{\mathrm{e}}^{\mathrm{T}},$$

$$\hat{\mathbf{S}}(\overset{+}{\mathbf{B}}_{\mathrm{e}}) = 2\frac{\partial\hat{\Psi}}{\partial\overset{+}{\mathbf{B}}_{\mathrm{e}}},$$

$$\overset{+}{\boldsymbol{\tau}} \equiv \overset{+}{\mathbf{R}}{}^{\mathrm{T}}\boldsymbol{\tau}\overset{+}{\mathbf{R}} = \overset{+}{\mathbf{V}}_{\mathrm{e}}\left[2\frac{\partial\hat{\Psi}}{\partial\overset{+}{\mathbf{B}}_{\mathrm{e}}}\right]\overset{+}{\mathbf{V}}_{\mathrm{e}}$$

$$= \overset{+}{\mathbf{B}}_{\mathrm{e}}\frac{\partial\hat{\Psi}}{\partial\overset{+}{\mathbf{B}}_{\mathrm{e}}} + \frac{\partial\hat{\Psi}}{\partial\overset{+}{\mathbf{B}}_{\mathrm{e}}}\overset{+}{\mathbf{B}}_{\mathrm{e}}$$

$$= 2\mathbb{S}_{\overset{+}{\mathbf{B}}_{\mathrm{e}}}\cdot\frac{\partial\hat{\Psi}}{\partial\overset{+}{\mathbf{B}}_{\mathrm{e}}}. \tag{20}$$

At this point, introduction of the elastic log-strain tensor

$$\overset{+}{\mathbf{H}}_{\mathrm{e}} \equiv \ln(\overset{+}{\mathbf{V}}_{\mathrm{e}}) = \tfrac{1}{2}\ln(\overset{+}{\mathbf{B}}_{\mathrm{e}}) \Leftrightarrow \overset{+}{\mathbf{V}}_{\mathrm{e}} = \exp(\overset{+}{\mathbf{H}}_{\mathrm{e}}) \quad \text{and} \quad \overset{+}{\mathbf{B}}_{\mathrm{e}} = \exp(2\overset{+}{\mathbf{H}}_{\mathrm{e}}) \tag{21}$$

makes a number of simplifications possible—particularly in the event that the energy function admits a quadratic representation. The essential results follow from the analysis contained in Section 2 and the Appendix of Dashner (1990). With repeated reference to these results, one need only make the identifications $\mathbf{a} = \overset{+}{\mathbf{H}}_{\mathrm{e}}$ and $\mathbf{b} = \overset{+}{\mathbf{B}}_{\mathrm{e}}$ and $(1/\rho)\mathbf{s}_{\mathrm{e}} = \overset{+}{\boldsymbol{\tau}}$ in order to conclude that

$$\overset{+}{\boldsymbol{\tau}} = \frac{\partial \hat{\Psi}}{\partial \overset{+}{\mathbf{H}}_{\mathrm{e}}}. \tag{22†}$$

Thus, in the event that strain energy is expressible as a quadratic in $\overset{+}{\mathbf{H}}_{\mathrm{e}}$ the "back-rotated" Kirchhoff stress is an isotropic linear function of this elastic log–strain tensor.

Rate or evolution equations are subject to similar simplifications. To see this, first observe that

$$\overset{+}{\mathbf{F}}_{\mathrm{e}} = \overset{+}{\mathbf{U}}\bar{\mathbf{U}}_{\mathrm{e}}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1},$$

$$\frac{\mathrm{d}\overset{+}{\mathbf{F}}_{\mathrm{e}}}{\mathrm{d}t} = \frac{\mathrm{d}\overset{+}{\mathbf{U}}}{\mathrm{d}t}\bar{\mathbf{U}}_{\mathrm{e}}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1} + \overset{+}{\mathbf{U}}\bar{\mathbf{U}}_{\mathrm{e}}\frac{\mathrm{d}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}}{\mathrm{d}t}$$

$$= \left(\frac{\partial\overset{+}{\mathbf{U}}}{\mathrm{d}t}\overset{+}{\mathbf{U}}^{-1}\right)(\overset{+}{\mathbf{U}}\bar{\mathbf{U}}_{\mathrm{e}}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}) + (\overset{+}{\mathbf{U}}\bar{\mathbf{U}}_{\mathrm{e}}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1})\left(\overset{+}{\mathbf{U}}_{\mathrm{p}}\frac{\mathrm{d}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}}{\mathrm{d}t}\right)$$

$$= \left(\frac{\mathrm{d}\overset{+}{\mathbf{U}}}{\mathrm{d}t}\overset{+}{\mathbf{U}}^{-1}\right)\overset{+}{\mathbf{F}}_{\mathrm{e}} - \overset{+}{\mathbf{F}}_{\mathrm{e}}\left(\frac{\mathrm{d}\overset{+}{\mathbf{U}}_{\mathrm{p}}}{\mathrm{d}t}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}\right),$$

$$\frac{\mathrm{d}\overset{+}{\mathbf{F}}_{\mathrm{e}}}{\mathrm{d}t} = \overset{+}{\mathbf{L}}\overset{+}{\mathbf{F}}_{\mathrm{e}} - \overset{+}{\mathbf{F}}_{\mathrm{e}}\mathbf{L}_{\mathrm{p}} \tag{23}$$

in terms of the velocity gradient tensors

$$\overset{+}{\mathbf{L}} \equiv \frac{\mathrm{d}\overset{+}{\mathbf{U}}}{\mathrm{d}t}\overset{+}{\mathbf{U}}^{-1}, \quad \mathbf{L}_{\mathrm{p}} \equiv \frac{\mathrm{d}\overset{+}{\mathbf{U}}_{\mathrm{p}}}{\mathrm{d}t}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}, \tag{24}$$

associated with the material flow in the $\overset{+}{\mathscr{B}}$ and $\mathscr{B}_{\mathrm{p}}$ configurations. As has been previously observed [*cf.* eqns (12), (13)], specification of the total strain rate $\dot{\mathbf{E}}$ is equivalent to the specification of $(\mathrm{d}\overset{+}{\mathbf{U}}/\mathrm{d}t)$, while specification of the plastic strain rate $\dot{\mathbf{E}}_{\mathrm{p}}$ is equivalent to the specification of $(\mathrm{d}\overset{+}{\mathbf{U}}_{\mathrm{p}}/\mathrm{d}t)$. In a similar manner, it follows from the associated deformation rate expressions

$$\overset{+}{\mathbf{D}} = (\overset{+}{\mathbf{L}})_{\mathrm{sym}} = \frac{1}{2}\left[\overset{+}{\mathbf{U}}^{-1}\frac{\mathrm{d}\overset{+}{\mathbf{U}}}{\mathrm{d}t} + \frac{\mathrm{d}\overset{+}{\mathbf{U}}}{\mathrm{d}t}\overset{+}{\mathbf{U}}^{-1}\right] = \mathbb{S}_{\overset{+}{\mathbf{U}}^{-1}}\cdot\frac{\mathrm{d}\overset{+}{\mathbf{U}}}{\mathrm{d}t},$$

$$\mathbf{D}_{\mathrm{p}} = (\mathbf{L}_{\mathrm{p}})_{\mathrm{sym}} = \frac{1}{2}\left[\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}\frac{\mathrm{d}\overset{+}{\mathbf{U}}_{\mathrm{p}}}{\mathrm{d}t} + \frac{\mathrm{d}\overset{+}{\mathbf{U}}_{\mathrm{p}}}{\mathrm{d}t}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}\right] = \mathbb{S}_{\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1}}\cdot\frac{\mathrm{d}\overset{+}{\mathbf{U}}_{\mathrm{p}}}{\mathrm{d}t} \tag{25}$$

and the invertibility of the symmetric fourth-order operator $\mathbb{S}$, that specification of these symmetric rate deformation rate functions is similarly equivalent. This in the sense that specification of $\overset{+}{\mathbf{D}}$ and $\mathbf{D}_{\mathrm{p}}$ determine the rates of change of the incremental stretch tensors $\overset{+}{\mathbf{U}}$ and $\overset{+}{\mathbf{U}}_{\mathrm{p}}$, as well as their associated antisymmetric spin tensors through the expressions

† *cf.* eqns (2.3) and (2.12) of Dashner (1990).

$$\frac{\mathrm{d}\overset{+}{\mathbf{U}}}{\mathrm{d}t} = \mathbb{S}_{\overset{+}{\mathbf{U}}}^{-1} \cdot \overset{+}{\mathbf{D}}, \quad \overset{+}{\mathbf{W}} \equiv \overset{+}{\mathbf{L}} - \overset{+}{\mathbf{D}} = \frac{\mathrm{d}\overset{+}{\mathbf{U}}}{\mathrm{d}t}\overset{+}{\mathbf{U}}^{-1} - \overset{+}{\mathbf{D}},$$

$$\frac{\mathrm{d}\overset{+}{\mathbf{U}}_{\mathrm{p}}}{\mathrm{d}t} = \mathbb{S}_{\overset{+}{\mathbf{U}}_{\mathrm{p}}}^{-1} \cdot \mathbf{D}_{\mathrm{p}}, \quad \mathbf{W}_{\mathrm{p}} \equiv \mathbf{L}_{\mathrm{p}} - \mathbf{D}_{\mathrm{p}} = \frac{\mathrm{d}\overset{+}{\mathbf{U}}_{\mathrm{p}}}{\mathrm{d}t}\overset{+}{\mathbf{U}}_{\mathrm{p}}^{-1} - \mathbf{D}_{\mathrm{p}}. \tag{26}$$

In addition, the evolution of the elastic deformation tensor $\overset{+}{\mathbf{B}}_{\mathrm{e}}$ is governed by the rate equation

$$\begin{aligned}
\frac{\mathrm{d}\overset{+}{\mathbf{B}}_{\mathrm{e}}}{\mathrm{d}t} &= \frac{\mathrm{d}\overset{+}{\mathbf{F}}_{\mathrm{e}}}{\mathrm{d}t}\overset{+}{\mathbf{F}}_{\mathrm{e}}^{\mathrm{T}} + \overset{+}{\mathbf{F}}_{\mathrm{e}}\left(\frac{\mathrm{d}\overset{+}{\mathbf{F}}_{\mathrm{e}}}{\mathrm{d}t}\right)^{\mathrm{T}} \\
&= (\overset{+}{\mathbf{L}}\overset{+}{\mathbf{F}}_{\mathrm{e}} - \overset{+}{\mathbf{F}}_{\mathrm{e}}\mathbf{L}_{\mathrm{p}})\overset{+}{\mathbf{F}}_{\mathrm{e}}^{\mathrm{T}} + \overset{+}{\mathbf{F}}_{\mathrm{e}}(\overset{+}{\mathbf{F}}_{\mathrm{e}}^{\mathrm{T}}\overset{+}{\mathbf{L}}^{\mathrm{T}} - \mathbf{L}_{\mathrm{p}}^{\mathrm{T}}\overset{+}{\mathbf{F}}_{\mathrm{e}}^{\mathrm{T}}) \\
&= \overset{+}{\mathbf{L}}\overset{+}{\mathbf{B}}_{\mathrm{e}} + \overset{+}{\mathbf{B}}_{\mathrm{e}}\overset{+}{\mathbf{L}}^{\mathrm{T}} - \overset{+}{\mathbf{F}}_{\mathrm{e}}(2\mathbf{D}_{\mathrm{p}})\overset{+}{\mathbf{F}}_{\mathrm{e}}^{\mathrm{T}}, \\
\frac{\overset{+}{\mathscr{D}}\overset{+}{\mathbf{B}}_{\mathrm{e}}}{\mathscr{D}t} &\equiv \frac{\mathrm{d}\overset{+}{\mathbf{B}}_{\mathrm{e}}}{\mathrm{d}t} + \overset{+}{\mathbf{B}}_{\mathrm{e}}\overset{+}{\mathbf{W}} - \overset{+}{\mathbf{W}}\overset{+}{\mathbf{B}}_{\mathrm{e}} \\
&= \overset{+}{\mathbf{D}}\overset{+}{\mathbf{B}}_{\mathrm{e}} + \overset{+}{\mathbf{B}}_{\mathrm{e}}\overset{+}{\mathbf{D}} - \overset{+}{\mathbf{F}}_{\mathrm{e}}\overset{+}{\mathbf{F}}_{\mathrm{e}}^{\mathrm{T}}(2\overset{+}{\mathbf{F}}_{\mathrm{e}}^{-\mathrm{T}}\mathbf{D}_{\mathrm{p}}\overset{+}{\mathbf{F}}_{\mathrm{e}}^{-1})\overset{+}{\mathbf{F}}_{\mathrm{e}}\overset{+}{\mathbf{F}}_{\mathrm{e}}^{\mathrm{T}} \\
&= \overset{+}{\mathbf{B}}_{\mathrm{e}}\overset{+}{\mathbf{D}} + \overset{+}{\mathbf{D}}\overset{+}{\mathbf{B}}_{\mathrm{e}} - \overset{+}{\mathbf{B}}_{\mathrm{e}}(2\overset{+}{\mathbf{F}}_{\mathrm{e}}^{-\mathrm{T}}\mathbf{D}_{\mathrm{p}}\overset{+}{\mathbf{F}}_{\mathrm{e}}^{-1})\overset{+}{\mathbf{B}}_{\mathrm{e}} \\
&= 2\mathbb{S}_{\overset{+}{\mathbf{B}}_{\mathrm{e}}} \cdot \overset{+}{\mathbf{D}} - 2\mathbb{B}_{\overset{+}{\mathbf{B}}_{\mathrm{e}}} \cdot \overset{+}{\mathbf{D}}_{\mathrm{p}}
\end{aligned} \tag{27}$$

written in terms of the objective corotational (Jaumann) derivative of $\overset{+}{\mathbf{B}}_{\mathrm{e}}$ relative to the material flow in $\overset{+}{\mathscr{B}}$, the material deformation rate $\overset{+}{\mathbf{D}}$ in $\overset{+}{\mathscr{B}}$ and the new plastic deformation rate

$$\overset{+}{\mathbf{D}}_{\mathrm{p}} \equiv \overset{+}{\mathbf{F}}_{\mathrm{e}}^{-\mathrm{T}}\mathbf{D}_{\mathrm{p}}\overset{+}{\mathbf{F}}_{\mathrm{e}}^{-1} \Leftrightarrow \mathbf{D}_{\mathrm{p}} \equiv \overset{+}{\mathbf{F}}_{\mathrm{e}}^{\mathrm{T}}\overset{+}{\mathbf{D}}_{\mathrm{p}}\overset{+}{\mathbf{F}}_{\mathrm{e}} \tag{28}$$

which represents the rate of material deformation in $\mathscr{B}_{\mathrm{p}}$—materially referred to $\overset{+}{\mathscr{B}}$. One final substitution results from the equality

$$\mathbb{S}_{\overset{+}{\mathbf{B}}_{\mathrm{e}}} \cdot \overset{+}{\mathscr{D}}_{\mathrm{p}} = \mathbb{B}_{\overset{+}{\mathbf{B}}_{\mathrm{e}}} \cdot \overset{+}{\mathbf{D}}_{\mathrm{p}}. \tag{29}$$

In view of the established properties of these fourth-order operators, this equation yields a unique solution for the "transformed" plastic deformation rate $\overset{+}{\mathscr{D}}_{\mathrm{p}}$, viz.

$$\overset{+}{\mathscr{D}}_{\mathrm{p}} = \mathbb{S}_{\overset{+}{\mathbf{B}}_{\mathrm{e}}}^{-1} \circ \mathbb{B}_{\overset{+}{\mathbf{B}}_{\mathrm{e}}} \cdot \overset{+}{\mathbf{D}}_{\mathrm{p}} = \mathbb{S}_{\overset{+}{\mathbf{B}}_{\mathrm{e}}^{-1}}^{-1} \cdot \overset{+}{\mathbf{D}}_{\mathrm{p}} \Leftrightarrow \overset{+}{\mathbf{D}}_{\mathrm{p}} = \mathbb{S}_{\overset{+}{\mathbf{B}}_{\mathrm{e}}^{-1}} \cdot \overset{+}{\mathscr{D}}_{\mathrm{p}}. \tag{30†}$$

† *cf.* property (iv), p. 322 of Dashner (1990).

Substitution of (29) into the rate expression (27) now yields the rate expressions

$$\frac{\overset{++}{\mathscr{D}}\mathbf{B}_{e}}{\mathscr{D}t} = 2\mathbb{S}_{\overset{+}{\mathbf{B}}_{e}}\cdot(\overset{+}{\mathbf{D}} - \overset{+}{\mathscr{D}}_{p}),$$

$$\frac{\overset{++}{\mathscr{D}}\mathbf{H}_{e}}{\mathscr{D}t} = [\![\partial\overset{+}{\mathbf{H}}_{e}/\partial\overset{+}{\mathbf{B}}_{e}]\!]\cdot\frac{\overset{++}{\mathscr{D}}\mathbf{B}_{e}}{\mathscr{D}t} = \{2[\![\partial\overset{+}{\mathbf{H}}_{e}/\partial\overset{+}{\mathbf{B}}_{e}]\!]\circ\mathbb{S}_{\overset{+}{\mathbf{B}}_{e}}\}\cdot(\overset{+}{\mathbf{D}} - \overset{+}{\mathscr{D}}_{p}), \tag{31}$$

for the elastic deformation and associated elastic log–strain tensor. In view of the results relating to the fourth-order tensor formed from the composition

$$\mathbb{H} \equiv 2[\![\partial\overset{+}{\mathbf{H}}_{e}/\partial\overset{+}{\mathbf{B}}_{e}]\!]\circ\mathbb{S}_{\overset{+}{\mathbf{B}}_{e}}, \tag{32}†$$

it follows that

$$\frac{\overset{++}{\mathscr{D}}\mathbf{H}_{e}}{\mathscr{D}t} = \overset{+}{\mathbf{D}} - \overset{+}{\mathscr{D}}_{p} + \tfrac{1}{3}[\overset{+}{\mathbf{H}}{}_{e}^{2}(\overset{+}{\mathbf{D}} - \overset{+}{\mathscr{D}}_{p}) - 2\overset{+}{\mathbf{H}}_{e}(\overset{+}{\mathbf{D}} - \overset{+}{\mathscr{D}}_{p})\overset{+}{\mathbf{H}}_{e} + (\overset{+}{\mathbf{D}} - \overset{+}{\mathscr{D}}_{p})\overset{+}{\mathbf{H}}{}_{e}^{2}] + \cdots,$$

$$\frac{\overset{++}{\mathscr{D}}\mathbf{H}_{e}}{\mathscr{D}t} \approx \overset{+}{\mathbf{D}} - \overset{+}{\mathscr{D}}_{p}, \quad |\overset{+}{\mathbf{H}}_{e}|^{2} \ll 1. \tag{33}$$

Thus one obtains a linear decomposition of the total deformation rate into purely elastic and purely plastic components, at least in the event that terms involving "squares" of elastic log–strain are negligible as compared to unity. It seems to me that (22) and $(33)_2$ fully reflect the simplifications to which Schieck and Stumpf refer.

4. CONCLUSION

In this note, an alternative kinematical formulation for the study of two superposed finite elastic–plastic deformations is proposed. It is claimed that this approach incorporates all essential physical features of this material model, is considerably less cumbersome, and would lead to the same or equivalent simplified forms that are realized by Schieck and Stumpf (1993). It must be noted however, that the simplified forms to which they refer are essentially "Eulerian" as opposed to "Lagrangean" in nature. In fact, since the only physically relevant deformation (insofar as energy and true stress are concerned) is the elastic deformation relating the current configuration to the elastically unstressed configuration, this problem is naturally amenable to the Eulerian modeling approach outlined in Dashner (1986b). Such an approach would, in my view, prove its worth for theoretical developments of this type.

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† *cf.* eqns (A28) and (A51), Appendix of Dashner (1990).

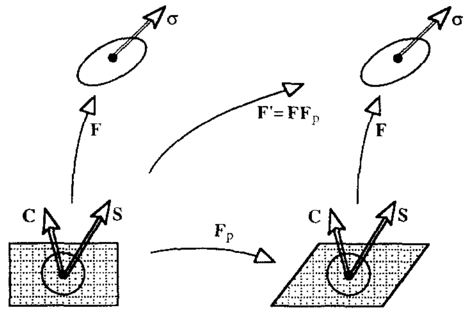

Fig. A1. Identical material response following pure "rotationless" plastic deformation.

APPENDIX. A SIMPLE ELASTO-PLASTIC, STRUCTURALLY ANISOTROPIC MODEL

Suppose an orthotropic monocrystalline solid for which permanent (or plastic) deformation results from the mechanism of pure slip along the active slip-planes within the crystal lattice. For this example it is assumed that such deformation takes place without *induced* effects such as material hardening or generation of backstress. Suppose also that the symmetric Piola–Kirchhoff stress $\mathbf{S}$ is determined as a function of the Green deformation tensor $\mathbf{C}$, viz.

$$\mathbf{S} = \mathscr{R}(\mathbf{C}), \tag{A1}$$

for all purely elastic deformations of the virgin specimen measured from the reference state. This equation implicitly contains all *descriptors* of the inherent structural anisotropy and is assumed to conform to an invariance relation

$$\mathbf{Q}\mathscr{R}(\mathbf{C})\mathbf{Q}^{T} = \mathscr{R}(\mathbf{QCQ}^{T})\ \forall\, \mathbf{Q} \in \mathscr{G}, \tag{A2}$$

expressed in terms of the orthogonal symmetry group $\mathscr{G}$ of the lattice in this initial rest configuration.

Now, for this simple model, it is clear than an internal element of a plastically deformed specimen at rest should be indistinguishable from an element taken from the virgin specimen—provided that their respective lattice structures are "aligned". Put differently, a pure plastic slip deformation which occurs without lattice rotation leads to a new state and configuration from which the same constitutive equations apply. Based on this observation it is a simple matter to construct the appropriate response equation (referenced to the initial configuration) which applies for a material element which is first subjected to a purely plastic "rotationless" slip deformation $\mathbf{F}_p$, followed by an elastic deformation $\mathbf{F}$ (see Fig. A1).

As measured from the initial configuration the (primed) deformation and stress measures

$$\begin{aligned}
\mathbf{F}' &= \mathbf{FF}_p,\\
\mathbf{C}' &= (\mathbf{F}')^{T}\mathbf{F}' = \mathbf{F}_p^{T}\mathbf{CF}_p, \quad \mathbf{C} \equiv \mathbf{F}^{T}\mathbf{F},\\
\mathbf{S}' &= \det(\mathbf{F}')(\mathbf{F}')^{-1}\boldsymbol{\sigma}(\mathbf{F}')^{-T}\\
&= \det(\mathbf{F})\mathbf{F}_p^{-1}\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}\mathbf{F}_p^{-T}, \quad \det(\mathbf{F}_p) = 1\\
&= \mathbf{F}_p^{-1}[\det(\mathbf{F})\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}]\mathbf{F}_p^{-T} = \mathbf{F}_p^{-1}[\mathbf{S}]\mathbf{F}_p^{-T}\\
&= \mathbf{F}_p^{-1}[\mathscr{R}(\mathbf{C})]\mathbf{F}_p^{-T}\\
&= \mathbf{F}_p^{-1}[\mathscr{R}(\mathbf{F}_p^{-T}\mathbf{C}'\mathbf{F}_p^{-1})]\mathbf{F}_p^{-T}
\end{aligned} \tag{A3}$$

are observed.

This expresses the proper relation between stress and strain for the material element as it is elastically deformed from the orthotropic rest state obtained from the reference state by the pure slip deformation (without lattice rotation) $\mathbf{F}_p$. Thus, by dropping the intermediate "prime" notation for the variables $\mathbf{S}$ and $\mathbf{C}$, the constitutive relation corresponding to this particular "evolved" state (but referred to the original reference) is given by

$$\mathbf{S} = \mathbf{F}_p^{-1}[\mathscr{R}(\mathbf{F}_p^{-T}\mathbf{CF}_p^{-1})]\mathbf{F}_p^{-T}. \tag{A4}$$

The central issue is now to ascertain whether this simple constitutive example falls within the scope of generality of the Green–Naghdi theory (Green and Naghdi, 1965)—as originally conceived.† That is, can this constitutive relation be recast in the (non-hardening) form

† This issue has been addressed in Dashner (1986a) and subsequently discussed by Casey (1987).

$$\mathbf{S} = \mathscr{R}''(\mathbf{E}, \mathbf{E}_p), \tag{A5}$$

or equivalently

$$\mathbf{S} = \mathscr{R}^*(\mathbf{C}, \mathbf{U}_p), \quad \mathbf{C} \equiv 2\mathbf{E} - \mathbf{I}, \quad \mathbf{U}_p \equiv \sqrt{2\mathbf{E}_p - \mathbf{I}}. \tag{A6}$$

By employing the polar decomposition

$$\mathbf{F}_p = \mathbf{R}_p\mathbf{U}_p \Rightarrow \mathbf{F}_p^{-1} = \mathbf{U}_p^{-1}\mathbf{R}_p^{T} \Rightarrow \mathbf{F}_p^{-T} = \mathbf{R}_p\mathbf{U}_p^{-1}, \tag{A7}$$

our "plastically-perturbed" constitutive equation (A4) takes the form

$$\mathbf{S} = \mathbf{U}_p^{-1}\mathbf{R}_p^{T}[\mathscr{R}(\mathbf{R}_p\mathbf{U}_p^{-1}\mathbf{C}\mathbf{U}_p^{-1}\mathbf{R}_p^{T})]\mathbf{R}_p\mathbf{U}_p^{-1}. \tag{A8}$$

With reference to the symmetry relation (A2) for the response function $\mathscr{R}$, it is clear that the rotational component $\mathbf{R}_p$ can be eliminated under the circumstance wherein $\mathbf{R}_p \in \mathscr{g}$. Given the kinematics of pure-slip deformation, there is no justification for supposing this to be true—unless of course we consider the degenerate case wherein the symmetry group $\mathscr{g}$ is the full orthogonal group. Alternatively, direct functional dependence of $\mathbf{R}_p$ on $\mathbf{U}_p$ (i.e. $\mathbf{R}_p = \mathscr{R}_p(\mathbf{U}_p)$) would also produce the desired form. This possibility must also be rejected as it is known that a given plastic deformation can result from a multiplicity of different slip-sequences producing different lattice orientations. An example of such a deformation sequence is given in Dashner (1986a). It is necessary to conclude, therefore, that *knowledge of the response function in the virgin reference state (with all of its inherent structural complexity) in addition to the present values of deformation* $\mathbf{C}$ *and plastic strain* $\mathbf{E}_p \Rightarrow \mathbf{U}_p$, *is not sufficient to fix the stress response in subsequent states as it evolves through commonly understood plastic deformation mechanisms.* With reference to the above response equation (A4), viz.

$$\mathbf{S} = \mathbf{F}_p^{-1}[\mathscr{R}(\mathbf{F}_p^{-T}\mathbf{C}\mathbf{F}_p^{-1})]\mathbf{F}_p^{-T}, \tag{A9}$$

this deficiency can be overcome by adopting a more precise definition for the non-singular tensor state variable $\mathbf{F}_p$. Observe that if $\mathbf{F}_p$ is *defined* as the pure plastic deformation (measured from the chosen reference) which places the current elastically unstressed element with lattice orientation identical to (or indistinguishable from) the virgin reference element, then state is fixed by specification of $\mathbf{C}$ and $\mathbf{F}_p$, and response is determined (in terms of the reference state response function $\mathscr{R}$) by the global response function

$$\mathbf{S} = \mathscr{S}(\mathbf{F}_p, \mathbf{C}) \equiv \mathbf{F}_p^{-1}[\mathscr{R}(\mathbf{F}_p^{-T}\mathbf{C}\mathbf{F}_p^{-1})]\mathbf{F}_p^{-T}. \tag{A10}$$

This definition extends the traditional one by ascribing causal significance to $\mathbf{F}_p$'s rotational component—thereby removing the oft discussed kinematical indeterminacy of the orientation of the so-called "unstressed" configuration. An appropriate evolution law for $\mathbf{F}_p$ based on considerations of lattice micromechanics and consistent with applicable invariance principles would be needed to complete the model. This approach, as I understand it, is not inconsistent with Mandel's notion of the "rigid triad" and reflects the thinking of a number of the other authors. In essence, it is a response to the understanding that the orientation of the invariant lattice structure in subsequent rest configurations is not a structural characteristic of the material—nor is it determined by the (induced) current value of the plastic strain. Indeed, the micromechanics of plastic deformation ensure that the evolution of lattice orientation (relative to the material) is fundamentally an induced effect of primary importance in its own right—even in the absence of other induced effects.

With regard to the application of invariance criteria and in light of the above definition, a superposed rigid body rotation ($\mathbf{Q}$) of the deformed material element effects the changes

$$\begin{aligned} \mathbf{F} &\to \mathbf{QF} \\ \mathbf{C} &\to \mathbf{C} \\ \mathbf{F}_p &\to \mathbf{F}_p, \end{aligned} \tag{A11}$$

while reorientation (prerotation) of the reference configuration ($\mathbf{P}$) is associated with the variable transformation

$$\begin{aligned} \mathbf{F} &\to \mathbf{FP}^T \\ \mathbf{C} &\to \mathbf{PCP}^T \\ \mathbf{F}_p &\to \mathbf{PF}_p\mathbf{P}^T. \end{aligned} \tag{A12}$$

This last relation serves to insure that the updated "unstressed" element maintains the same lattice orientation as the transformed reference element. With these "prerotation" transformation relations and the symmetry property of the reference state response function $\mathscr{R}$, it is a simple matter to verify the induced symmetry relation for the global response function (A10), viz.

$$\mathscr{S}(\mathbf{PF}_p\mathbf{P}^T, \mathbf{PCP}^T) = \mathbf{P}[\mathscr{S}(\mathbf{F}_p, \mathbf{C})]\mathbf{P}^T \;\forall\, \mathbf{P} \in \mathscr{g}, \tag{A13}$$

expressed in terms of the orthogonal symmetry group $\mathscr{g}$ of the material element in its reference state. Moreover, for any orthogonal $\mathbf{M} \in \mathscr{g}$, it follows that

$$
\begin{aligned}
\mathcal{S}(\mathbf{F}_p, \mathbf{C}) &= \mathbf{F}_p^{-1}[\mathscr{R}(\mathbf{F}_p^{-T}\mathbf{C}\mathbf{F}_p^{-1})]\mathbf{F}_p^{-T} \\
&= \mathbf{F}_p^{-1}\mathbf{M}^T[\mathscr{R}(\mathbf{M}\mathbf{F}_p^{-T}\mathbf{C}\mathbf{F}_p^{-1}\mathbf{M}^T)]\mathbf{M}\mathbf{F}_p^{-T}, \quad \mathbf{M} \in \mathscr{g} \\
&= (\mathbf{M}\mathbf{F}_p)^{-1}\{\mathscr{R}[(\mathbf{M}\mathbf{F}_p)^{-T}\mathbf{C}(\mathbf{M}\mathbf{F}_p)^{-1}]\}(\mathbf{M}\mathbf{F}_p)^{-T} \\
\mathcal{S}(\mathbf{F}_p, \mathbf{C}) &= \mathcal{S}(\mathbf{M}\mathbf{F}_p, \mathbf{C}) \; \forall \, \mathbf{M} \in \mathscr{g}.
\end{aligned} \tag{A14}
$$

In view of the above causal definition of $\mathbf{F}_p$, this is readily interpreted by observing that if $\mathbf{F}_p$ places the unstressed material element with orientation indistinguishable from the virgin reference element, then so also does $\mathbf{M}\mathbf{F}_p$ for any $\mathbf{M} \in \mathscr{g}$. This may be concisely stated in terms of partial invariance under post-rotation ($\mathbf{M}$) of the intermediate configuration and the associated transformations

$$
\left.
\begin{aligned}
\mathbf{F} &\rightarrow \mathbf{F} \\
\mathbf{C} &\rightarrow \mathbf{C} \\
\mathbf{F}_p &\rightarrow \mathbf{M}\mathbf{F}_p \\
\mathbf{F}_e = \mathbf{F}\mathbf{F}_p^{-1} &\rightarrow \mathbf{F}_e\mathbf{M}^T
\end{aligned}
\right\} \forall \, \mathbf{M} \in \mathscr{g}. \tag{A15}
$$

It is worth reiterating that for the case of full structural isotropy, the elastic response equations are completely insensitive to arbitrary specification of $\mathbf{M}$, and thus to the left rotational component $\mathbf{F}_p = \mathbf{R}_p\mathbf{U}_p$ of the plastic deformation gradient, and that the response equations themselves are fully isotropic functions of their tensor arguments. Taken together, full structural isotropy ensures that the symmetric Piola–Kirchhoff stress $\mathbf{S}$ is expressible as a fully isotropic function of the deformation measures $\mathbf{U}_p$ and $\mathbf{C}$, or equivalently $\mathbf{E}_p$ and $\mathbf{E}$, in accordance with the Green–Naghdi forms. Put differently, the elastic strain energy and stress response functions for such a material are expressible as

$$
\begin{aligned}
\Psi &= \hat{\Psi}(\mathbf{C}_e) \\
\mathbf{S} &= \mathbf{F}_p^{-1}\mathscr{R}(\mathbf{C}_e)\mathbf{F}_p^{-T}
\end{aligned}
\quad \mathbf{C}_e \equiv \mathbf{F}_e^T\mathbf{F}_e = \mathbf{F}_p^{-T}\mathbf{C}\mathbf{F}_p^{-1} \tag{A16}
$$

in terms of isotropic scalar and tensor-valued functions $\hat{\Psi}$ and $\mathscr{R}$, and *any* kinematically compatible gradient decomposition

$$
\mathbf{F} = \mathbf{F}_e\mathbf{F}_p. \tag{A17}
$$

Since Kirchhoff stress $\boldsymbol{\tau} = (\rho_o/\rho)\,\boldsymbol{\sigma}$ is given by

$$
\boldsymbol{\tau} = \mathbf{F}\mathbf{S}\mathbf{F}^T, \tag{A18}
$$

it generally follows that

$$
\begin{aligned}
\boldsymbol{\tau} &= \mathbf{F}_e\mathbf{F}_p[\mathbf{F}_p^{-1}\mathscr{R}(\mathbf{C}_e)\mathbf{F}_p^{-T}]\mathbf{F}_p^T\mathbf{F}_e^T \\
&= \mathbf{F}_e[\mathscr{R}(\mathbf{F}_e^T\mathbf{F}_e)]\mathbf{F}_e^T \\
&= \mathbf{R}_e\{\mathbf{U}_e[\mathscr{R}(\mathbf{U}_e^2)]\mathbf{U}_e\}\mathbf{R}_e^T, \quad \mathbf{F}_e = \mathbf{R}_e\mathbf{U}_e
\end{aligned} \tag{A19}
$$

in terms of the polar components of the elastic gradient $\mathbf{F}_e$. This form is subject to invariance under the group transformation (A15). For the fully isotropic case, the left-handed polar decomposition

$$
\mathbf{F}_e = \mathbf{V}_e\mathbf{R}_e \Leftrightarrow \mathbf{V}_e = \sqrt{\mathbf{F}_e\mathbf{F}_e^T} \tag{A20}
$$

results in the well-known isotropic form

$$
\boldsymbol{\tau} = \mathbf{V}_e[\mathscr{R}(\mathbf{V}_e^2)]\mathbf{V}_e. \tag{A21}
$$